

STOCHASTIC PARABOLIC EQUATIONS OF FULL SECOND ORDER

S. V. LOTOTSKY AND B. L. ROZOVSKII

ABSTRACT. A procedure is described for defining a generalized solution for stochastic differential equations using the Cameron-Martin version of the Wiener Chaos expansion. Existence and uniqueness of this Wiener Chaos solution is established for parabolic stochastic PDEs such that both the drift and the diffusion operators are of the second order.

1. INTRODUCTION

Consider a stochastic evolution equation

$$(1.1) \quad du(t) = (\mathcal{A}u(t) + f(t))dt + (\mathcal{M}u(t) + g(t))dW(t),$$

where \mathcal{A} and \mathcal{M} are differential operators, and W is a Wiener process on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Traditionally, this equation was studied under the following assumptions:

- **A1.** The operator \mathcal{A} is elliptic, the order of the operator \mathcal{M} is less than the order of \mathcal{A} , and $\mathcal{A} - \frac{1}{2}\mathcal{M}\mathcal{M}^*$ is elliptic (possibly degenerate) operator,

In fact, it is well known that unless assumption **A1** holds, equation (1.1) has no solutions in $L_2(\Omega; X)$ for any reasonable choice of the state space X .

It was shown recently (see [4, 5, 6] and the references therein) that if only the operator \mathcal{A} is elliptic and the order of \mathcal{M} is *smaller* than the order of \mathcal{A} , then there exists a unique generalized solution of equation (1.1). This solution is often referred to as Wiener Chaos solution. It is given by the Wiener chaos expansion $u(t) = \sum_{|\alpha| < \infty} u_\alpha(t) \xi_\alpha$, where $\{\xi_\alpha\}_{|\alpha| < \infty}$ is the Cameron-Martin orthonormal basis in the space $L_2(\Omega; \mathcal{F}^W; X)$ of square integrable random elements in X measurable with respect to the sigma-algebra \mathcal{F}^W generated by the Wiener process. The Cameron-Martin basis $\{\xi_\alpha\}$ is indexed by multiindices $\alpha = (\alpha_1, \alpha_2, \dots)$. It was shown that for certain positive weights $Q = \{q(\alpha)\}_{|\alpha| < \infty}$, the weighted norm

$$\|u\|_{Q,X}^2 := \sum_{|\alpha| < \infty} q^2(\alpha) \|u_\alpha\|_{L_2((0,T);X)}^2 < \infty,$$

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where X is the appropriate Hilbert space characterizing the “regularity” of the solution. Note that without assumption **A1**

$$E \|u\|_{L_2((0,T);X)}^2 = \sum_{|\alpha| < \infty} \|u_\alpha(t)\|_{L_2((0,T);X)}^2 = \infty.$$

In this paper, we consider the Cauchy problem for the following stochastic partial differential equation:

$$(1.2) \quad du = (a_{ij}D_iD_ju + b_iD_iu + cu + f)dt + (\rho_{ij}D_iD_ju + \sigma_iD_iu + \nu u + g)dW, \quad t \in [0, T], \quad x \in \mathbb{R}.$$

In contrast to the previous work, this is a parabolic SPDE of the *full* second order, in that the drift and diffusion operators have the same order 2. We construct a scale of weighted Wiener chaos spaces (related but not identical to Kondratiev’s spaces) and prove the existence and uniqueness of the solution in the spaces from this scale.

2. CONSTRUCTING A SOLUTION: AN EXAMPLE

Let $\mathbb{F} = (\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0 \leq t \leq T}, \mathbb{P})$ be a stochastic basis with the usual assumptions and $W = W(t)$, $0 \leq t \leq T$, a standard Wiener process on \mathbb{F} . For a Hilbert space X , denote by $L_2(\mathbb{W}; X)$ the collection of X -valued random elements that are square integrable ($\mathbb{E}\|\cdot\|_X^2 < \infty$) and are measurable with respect to the sigma-algebra generated by $W(t)$, $t \in [0, T]$.

Consider the Itô equation

$$(2.1) \quad u(t, x) = e^{-x^2/2} + \int_0^t u_{xx}(s, x)ds + \int_0^t u_{xx}(s, x)dW(s), \quad t \in [0, T], \quad x \in \mathbb{R}.$$

If there is a solution, its Fourier transform in space, $\hat{u}(t, y) = (1/\sqrt{2\pi}) \int_{\mathbb{R}} e^{-ixy}u(t, x)dx$ satisfies

$$(2.2) \quad \hat{u}(t, y) = e^{-y^2/2} - y^2 \int_0^t \hat{u}(s, y)ds - y^2 \int_0^t \hat{u}(s, y)dW(s), \quad t \in [0, T], \quad y \in \mathbb{R}.$$

For each fixed y , (2.2) defines a geometric Brownian motion:

$$(2.3) \quad \hat{u}(t, y) = e^{-(1+t)y^2 - (y^4/2)t - y^2W(t)}.$$

Let $H^\gamma(\mathbb{R})$ be the Sobolev space

$$(2.4) \quad \left\{ f : \int_{\mathbb{R}} (1 + |y|^2)^\gamma |\hat{f}(y)|^2 dy < \infty \right\}.$$

Since

$$(2.5) \quad \mathbb{E}|\hat{u}(t, y)|^2 = e^{-2(1+t)y^2 + y^4t},$$

the solution of (2.1) cannot be an element of $L_2(\mathbb{W}; L_2((0, T); H^\gamma(\mathbb{R})))$ for *any* $\gamma \in \mathbb{R}$, even though the initial condition is non-random and is an element of $H^\gamma(\mathbb{R})$ for *every* $\gamma \in \mathbb{R}$.

Let us try another approach. Once again, assuming that the solution exists, we apply the Itô formula to the product $u(t, x)\mathcal{E}_h(t)$, where

$$(2.6) \quad \mathcal{E}_h(t) = \exp \left(\int_0^t h(s) dW(s) - \frac{1}{2} \int_0^t h^2(s) ds \right)$$

and $h = h(t)$ is a smooth deterministic function. Since

$$(2.7) \quad \mathcal{E}_h(t) = 1 + \int_0^t \mathcal{E}_h(s) h(s) dW(s),$$

we conclude that the function

$$(2.8) \quad u_h(t, x) = \mathbb{E}(u(t, x)\mathcal{E}_h(t)),$$

if defined, must satisfy the heat equation

$$(2.9) \quad u_h(t, x) = e^{-x^2/2} + \int_0^t (1 + h(s)) \frac{\partial^2 u_h(s, x)}{\partial x^2} ds.$$

If $\sup_t |h(t)| < 1$, then this equation has a unique solution in every $H^\gamma(\mathbb{R})$ and

$$(2.10) \quad u_h(t, x) = \mathbb{E} \exp(-(X(t, x))^2/2),$$

where

$$(2.11) \quad X(t, x) = x + \int_0^t \sqrt{2(1 + h(s))} dW(s).$$

In other words, while existence of a solution of equation (2.1) is still unclear, we now have a family of functions $u_h(t, x)$ defined by (2.10). All we need now is a systematic procedure of relating the family of deterministic functions $u_h = u_h(t, x)$ to a random process $u = u(t, x)$; then this process is natural to call a solution of (2.1).

Here is a possible way of constructing a stochastic process from u_h . Let $\mathbf{m} = \{m_k, k \geq 1\}$ be the Fourier cosine basis in $L_2((0, T))$:

$$(2.12) \quad m_1(s) = \frac{1}{\sqrt{T}}; \quad m_k(t) = \sqrt{\frac{2}{T}} \cos \left(\frac{\pi(k-1)t}{T} \right), \quad k > 1; \quad 0 \leq t \leq T.$$

Then

$$(2.13) \quad h(t) = \sum_{k \geq 1} h_k m_k(t),$$

For every fixed $t \in [0, T]$ and $\gamma \in \mathbb{R}$, we can now interpret the function $u_h(t, \cdot)$ as a mapping from the set of sequences $h = (h_1, h_2, \dots)$ to the space $H^\gamma(\mathbb{R}^d)$, and, as equalities (2.10) and (2.11) suggest, this mapping is analytic in the region $\{h : \sum_{k \geq 1} h_k^2 < \varepsilon\}$ for sufficiently small ε . We will now compute the derivatives of this mapping.

Let \mathcal{J} be the collection of multi-indices $\alpha = \{\alpha_k, k \geq 1\}$. Each $\alpha \in \mathcal{J}$ has non-negative integer elements α_k and

$$(2.14) \quad |\alpha| = \sum_k \alpha_k < \infty.$$

We also use the notation

$$(2.15) \quad \alpha! = \prod_k \alpha_k!$$

and consider special multi-indices, $\alpha = (0)$ with $|\alpha| = 0$ and $\alpha = \varepsilon_i$, with $|\alpha| = 1$, $\alpha_i = 1$.

For each $\alpha \in \mathcal{J}$ define

$$(2.16) \quad u_\alpha(t, x) = \frac{1}{\sqrt{\alpha!}} \frac{\partial^{|\alpha|} u_h(t, x)}{\partial h_1^{\alpha_1} \partial h_2^{\alpha_2} \dots} \Big|_{h=0}.$$

Then

$$(2.17) \quad u_h(t, x) = \sum_{\alpha \in \mathcal{J}} u_\alpha(t, x) \frac{h^\alpha}{\sqrt{\alpha!}},$$

where

$$(2.18) \quad h^\alpha = \prod_{k \geq 1} h_k^{\alpha_k}.$$

On the other hand, by direct computation,

$$(2.19) \quad \mathcal{E}_h(t) = \mathbb{E}(\mathcal{E}_h(T) | \mathcal{F}_t^W) = \sum_{\alpha \in \mathcal{J}} \frac{h^\alpha}{\sqrt{\alpha!}} \xi_\alpha(t),$$

where

$$(2.20) \quad \xi_\alpha(t) = \mathbb{E}(\xi_\alpha | \mathcal{F}_t^W), \quad \xi_\alpha = \frac{1}{\sqrt{\alpha!}} \prod_{k \geq 1} H_{\alpha_k} \left(\int_0^T m_k(t) dW(t) \right),$$

and

$$(2.21) \quad H_n(x) = (-1)^n e^{x^2/2} \frac{d^n}{dx^n} e^{-x^2/2}$$

is n -th Hermite polynomial. It is a standard fact [1] that the collection $\{\xi_\alpha, \alpha \in \mathcal{J}\}$ is an orthonormal basis in $L_2(\mathbb{W}; \mathbb{R})$.

The functions $u_\alpha(t, x)$, $\alpha \in \mathcal{J}$, uniquely determine $u_h(t, x)$ according to (2.17). On the other hand, if

$$(2.22) \quad \sum_{\alpha \in \mathcal{J}} \|u_\alpha(t)\|_{H^\gamma(\mathbb{R})}^2 < \infty,$$

then the $H^\gamma(\mathbb{R})$ -valued random process

$$(2.23) \quad u(t, x) = \sum_{\alpha \in \mathcal{J}} u_\alpha(t, x) \xi_\alpha$$

satisfies $\mathbb{E}(u(t, x) \mathcal{E}_h(t)) = u_h(t, x)$; if, in addition, u is \mathcal{F}_t^W -adapted, then also

$$(2.24) \quad u(t, x) = \sum_{\alpha \in \mathcal{J}} u_\alpha(t, x) \xi_\alpha(t).$$

If condition (2.22) fails, then (2.23) is a formal series, which we *define* to be the stochastic process corresponding to the family u_h .

As (2.5) suggests, if u_h is the solution of (2.9), then (2.22) fails for every γ . Let us now see how fast the series diverges. Equality (2.9) implies

$$\begin{aligned}
 (2.25) \quad & u_{(0)}(t, x) = e^{-x^2/2} + \int_0^t \frac{\partial^2 u_{(0)}(s, x)}{\partial x^2} ds, \quad |\alpha| = 0; \\
 & u_{\epsilon_i}(t) = \int_0^t \frac{\partial^2 u_{\epsilon_i}(s, x)}{\partial x^2} ds + \int_0^t \frac{\partial^2 u_{(0)}(s, x)}{\partial x^2} m_i(s) ds, \quad |\alpha| = 1; \\
 & u_\alpha(t) = \int_0^t \frac{\partial^2 u_\alpha(s, x)}{\partial x^2} ds + \sum_{k=1}^{\infty} \sqrt{\alpha_k} \int_0^t \frac{\partial^2 u_{\alpha - \epsilon_k}(s, x)}{\partial x^2} m_k(s) ds, \quad |\alpha| > 1.
 \end{aligned}$$

Equations of the type (2.25) have been studied [4, Section 6 and References]. In particular, it is known that

$$(2.26) \quad \sum_{|\alpha|=n} \|u_\alpha(t)\|_{H^\gamma(\mathbb{R})}^2 = \frac{t^n}{n!} \|D_x^{2n} \Phi_t u_0\|_{H^\gamma(\mathbb{R})}^2,$$

where $D_x = \partial/\partial x$, Φ_t is the heat semigroup, and $u_0(x) = e^{-x^2/2}$. To simplify further computation, let us assume that $\gamma = 0$. Then, switching to the Fourier transform,

$$(2.27) \quad \|D_x^{2n} \Phi_t u_0\|_{L_2(\mathbb{R}^d)}^2 = \int_{\mathbb{R}} |y|^{4n} e^{-y^2(t+1)} dy = \frac{\Gamma(2n + \frac{1}{2})}{(1+t)^{2n}}.$$

Using Stirling's formula for the Gamma function Γ ,

$$(2.28) \quad \sum_{|\alpha|=n} \|u_\alpha(t)\|_{L_2(\mathbb{R})}^2 = \left(\frac{2\sqrt{t}}{1+t} \right)^{2n} C(n)n!,$$

where the numbers $C(n)$ are uniformly bounded from above and below. Similar result holds in every $H^\gamma(\mathbb{R})$. Thus, (2.22) does not hold, but instead, by (2.28), we have

$$(2.29) \quad \sum_{\alpha \in \mathcal{J}} \frac{1}{|\alpha|!} \|u_\alpha(t)\|_{H^\gamma(\mathbb{R})}^2 < \infty.$$

We denote by $(\mathfrak{L})_{0,0}(\mathbb{W}; H^\gamma(\mathbb{R}))$ the collection of formal series (2.24) satisfying (2.29); the reason for using $(\mathfrak{L})_{0,0}$ in the notation will become clear later. Note that we had equalities in all computations for equation (2.1) that lead to (2.29), which suggests that $(\mathfrak{L})_{0,0}(\mathbb{W}; H^\gamma(\mathbb{R}))$ is the natural solution space for equation (2.1). For a more general stochastic parabolic equation of full second order in \mathbb{R}^d , the natural solution space turns out to be $(\mathfrak{L})_{p,q}(\mathbb{W}; L_2((0, T); H^\gamma(\mathbb{R}^d)))$ for suitable $p, q \leq 0$.

In the next section we address the following questions:

- (1) How to define the spaces $(\mathfrak{L})_{p,q}(\mathbb{W}; X)$ for $p, q \in \mathbb{R}$ without relying on an orthonormal basis in $L_2((0, T))$?
- (2) How to construct a solution of a general stochastic parabolic equations of full second order?

3. GENERAL CONSTRUCTIONS AND THE MAIN RESULT

As before, let $\mathbb{F} = (\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0 \leq t \leq T}, \mathbb{P})$ be a stochastic basis with the usual assumptions and $W = W(t)$, $0 \leq t \leq T$, a standard Wiener process on \mathbb{F} . Denote by $H^s = H^s((0, T))$, $s \geq 0$, the Sobolev spaces on $(0, T)$ with norm

$$(3.1) \quad \|\cdot\|_s = \|\Lambda^{s/2} \cdot\|_{L_2((0, T))},$$

where Λ is the operator

$$(3.2) \quad 1 - \frac{T^2}{\pi^2} \frac{d^2}{dt^2}$$

with Neumann boundary conditions. This norm extends to functions of several variables via the tensor product of the spaces H^s .

Definition 3.1. *Given real numbers p, q and a Hilbert space X , $(\mathfrak{L})_{p,q}(\mathbb{W}; X)$ is the closure of the set of X -valued random elements*

$$(3.3) \quad \eta = \eta_0 + \sum_{k=1}^N \int_0^T \int_0^{s_k} \dots \int_0^{s_2} \eta_k(s_1, \dots, s_k) dW(s_1) \dots dW(s_{k-1}) dW(s_k), \quad N \geq 1,$$

with respect to the norm

$$(3.4) \quad \|\eta\|_{p,q;X}^2 = \|\eta_0\|_X^2 + \sum_{k=1}^N \frac{2^{kp}}{(k!)^2} \|\eta_k\|_q^2,$$

where each η_k , $k \geq 1$, is a smooth symmetric function from $[0, T]^k$ to X .

Remark 3.2. (a) *It is known [2, 7] that, for η of the type (3.3),*

$$(3.5) \quad \mathbb{E}\|\eta\|_X^2 = \|\eta_0\|_X^2 + \sum_{k=1}^N \frac{1}{k!} \|\eta_k\|_0^2.$$

(b) *The definition of each individual $(\mathfrak{L})_{p,q}(\mathbb{W}; X)$ inevitably involves arbitrary choices, such as the norm in $H^q((0, T))$. Further analysis shows that different choices result in shifts of the indices p, q , and the space $\cup_{p,q}(\mathfrak{L})_{p,q}(\mathbb{W}; X)$ does not depend on any arbitrary choices. In the white noise setting, where Ω is the space $\mathcal{S}'(\mathbb{R}^d)$ of the Schwartz distributions and \mathbb{P} is the normalized Gaussian measure on \mathcal{S} , the inductive limit $\cup_{p,q}(\mathfrak{L})_{p,q}(\mathbb{W}; \mathbb{R})$ is the Kondratiev space $(\mathcal{S})_{-1}$ [2].*

Remark 3.3. *If X is the Sobolev space $H^\gamma(\mathbb{R}^d)$, then we denote the norm $\|\cdot\|_{p,q;X}$ by $\|\cdot\|_{p,q;\gamma}$:*

$$(3.6) \quad \|\cdot\|_{p,q;H^\gamma(\mathbb{R}^d)} = \|\cdot\|_{p,q;\gamma}.$$

Proposition 3.4. *Let $\eta = f\xi_\alpha$, where $f \in X$ and ξ_α is defined by (2.20). Then*

$$(3.7) \quad \|\eta\|_{p,q;X}^2 = \frac{2^{|\alpha|p}}{|\alpha|!} \mathbb{N}^{2q\alpha} \|f\|_X^2,$$

where

$$(3.8) \quad \mathbb{N}^{2q\alpha} = \prod_{k \geq 1} k^{2q\alpha_k}.$$

Proof. Let $|\alpha| = n$. It is known [3] that

$$(3.9) \quad \xi_\alpha = \frac{1}{\sqrt{\alpha!}} \int_0^T \int_0^{s_n} \dots \int_0^{s_2} E_\alpha(s_1, \dots, s_n) dW(s_1) \dots dW(s_{n-1}) dW(s_n),$$

where E_α is the symmetric function

$$(3.10) \quad E_\alpha(s_1, \dots, s_n) = \sum_{\sigma \in \mathcal{P}_n} m_{i_1}(s_{\sigma(1)}) \dots m_{i_n}(s_{\sigma(n)}).$$

In (3.10), the summation is over all permutations of $\{1, \dots, n\}$, the functions m_k are defined in (2.12), and the positive integer numbers $i_1 \leq i_2 \leq \dots \leq i_n$ are such that, for every sequence $(b_k, k \geq 1)$ of positive numbers,

$$(3.11) \quad \prod_{k \geq 1} b_k^{\alpha_k} = b_{i_1} \cdot b_{i_2} \cdot \dots \cdot b_{i_n}.$$

For example, if $\alpha = (1, 0, 2, 0, 0, 4, 0, 0, \dots)$, then $|\alpha| = 7$ and $i_1 = 1, i_2 = i_3 = 3, i_4 = \dots = i_7 = 6$. Thus, in the notations of (3.4), we have

$$(3.12) \quad \eta_k = \begin{cases} \frac{1}{\sqrt{\alpha!}} E_\alpha f, & \text{if } k = n, \\ 0, & \text{otherwise} \end{cases}$$

Note that

$$(3.13) \quad \|E_\alpha\|_0 = \sqrt{\alpha!} \sqrt{n!}.$$

By definition (3.2) of the operator Λ we have

$$(3.14) \quad \Lambda^{q/2} m_k = k^q m_k \quad \text{or} \quad \|E_\alpha\|_q^2 = \mathbb{N}^{2q\alpha} \alpha! n!.$$

The result now follows. □

Corollary 3.5. *A formal series*

$$(3.15) \quad \eta = \sum_{\alpha \in \mathcal{J}} \eta_\alpha \xi_\alpha,$$

with $\eta_\alpha \in X$, is an element of $(\mathcal{L})_{p,q}(\mathbb{W}; X)$ if and only if

$$(3.16) \quad \sum_{\alpha \in \mathcal{J}} \frac{2^{p|\alpha|} \mathbb{N}^{2q\alpha}}{|\alpha!|} \|\eta_\alpha\|_X^2 < \infty.$$

Proof. This follows from (3.14) and the equality

$$(3.17) \quad \|E_\alpha + E_\beta\|_0^2 = \|E_\alpha\|_0^2 + \|E_\beta\|_0^2, \quad \alpha \neq \beta.$$

□

Denote by $(\mathcal{L})^{p,q}(\mathbb{W})$ the Hilbert space dual of $(\mathcal{L})_{p,q}(\mathbb{W}; \mathbb{R})$ relative to the inner product in $L_2(\mathbb{W}; \mathbb{R})$, and by $\langle \cdot, \cdot \rangle$ the corresponding duality. In the white noise setting, $\cap_{p,q} (\mathcal{L})^{p,q}(\mathbb{W})$ is the space $(\mathcal{S})_1$ of the Kondratiev test functions [2]. If $\eta \in (\mathcal{L})_{p,q}(\mathbb{W}; X)$ and $\zeta \in (\mathcal{L})^{p,q}(\mathbb{W})$, then $\langle \eta, \zeta \rangle$ is defined and belongs to X .

For $h \in L_2((0, T))$, define

$$(3.18) \quad \mathcal{E}_h = \mathcal{E}_h(T) = \exp \left(\int_0^T h(s) dW(s) - \frac{1}{2} \int_0^T |h(s)|^2 ds \right).$$

Proposition 3.6. *The random variable \mathcal{E}_h is an element of $(\mathfrak{L})^{p,q}(\mathbb{W})$ if and only if*

$$(3.19) \quad \|h\|_{-q}^2 < 2^p.$$

Proof. Since

$$(3.20) \quad \mathcal{E}_h(T) = 1 + \int_0^T h(t) \mathcal{E}_h(t) dt,$$

it follows that

$$(3.21) \quad \mathcal{E}_h = 1 + \sum_{k=1}^{\infty} \int_0^T \int_0^{s_k} \dots \int_0^{s_2} h(s_k) \dots h(s_1) dW(s_1) \dots dW(s_{k-1}) dW(s_k).$$

By (3.4) and (3.5), $\mathcal{E}_h \in (\mathfrak{L})^{p,q}(\mathbb{W})$ if and only if

$$(3.22) \quad \sum_{k=1}^{\infty} \left(2^{-p} \|h\|_{-q}^2 \right)^k < \infty,$$

that is, $\|h\|_{-q}^2 < 2^p$. □

Definition 3.7. *We say that the function h is sufficiently small if (3.19) holds for sufficiently large (positive) $-p, -q$.*

Proposition 3.8. *If $u \in \bigcup_{p,q} (\mathfrak{L})_{p,q}(\mathbb{W}; X)$ and h is sufficiently small, then*

$$(3.23) \quad u_h = \langle\langle u, \mathcal{E}_h \rangle\rangle$$

is an X -valued analytic function of h .

Proof. For every $u \in \bigcup_{p,q} (\mathfrak{L})_{p,q}(\mathbb{W}; X)$, there exist p, q such that $u \in (\mathfrak{L})_{p,q}(\mathbb{W}; X)$; by Proposition 3.6, u_h will indeed be defined for sufficiently small h . Similar to (2.17) we have

$$(3.24) \quad u_h = \sum_{\alpha \in \mathcal{J}} \frac{u_{\alpha} h^{\alpha}}{\sqrt{\alpha!}}$$

and this power series in h^{α} converges in some (infinite-dimensional) neighborhood of zero. □

From now on, $D_i = \partial/\partial x_i$, and the summation convention is in force: $c_i d_i = \sum_i c_i d_i$, etc.

Consider the linear equation in \mathbb{R}^d

$$(3.25) \quad du = (a_{ij} D_i D_j u + b_i D_i u + cu + f) dt + (\rho_{ij} D_i D_j u + \sigma_i D_i u + \nu u + g) dW$$

with initial condition $u(0, x) = v(x)$, under the following **assumptions**:

B0 All coefficients are non-random.

B1 The functions $a_{ij} = a_{ij}(t, x)$, $\rho_{ij} = \rho_{ij}(t, x)$ are measurable and bounded in (t, x) by a positive number C_0 , and
(i)

$$|a_{ij}(t, x) - a_{ij}(t, y)| + |\rho_{ij}(t, x) - \rho_{ij}(t, y)| \leq C_0|x - y|, \quad x, y \in \mathbb{R}^d, \quad 0 \leq t \leq T;$$

(ii) the matrix (a_{ij}) is uniformly positive definite, that is, there exists a $\delta > 0$ so that, for all vectors $y \in \mathbb{R}^d$ and all (t, x) , $a_{ij}y_iy_j \geq \delta|y|^2$.

B2 The functions $b_i = b_i(t, x)$, $c = c(t, x)$, $\sigma_i = \sigma_i(t, x)$, and $\nu = \nu(t, x)$ are measurable and bounded in (t, x) by the number C_0 .

B2

$$(3.26) \quad u_0 \in \bigcup_{p,q} (\mathfrak{L})_{p,q}(\mathbb{W}; L_2(\mathbb{R}^d)), \quad f, g \in \bigcup_{p,q} (\mathfrak{L})_{p,q}(\mathbb{W}; L_2((0, T); H^{-1}(\mathbb{R}^d))).$$

For simplicity, we introduce the following notations for the differential operators in (3.25):

$$(3.27) \quad \mathcal{A} = a_{ij}D_iD_j + b_iD_i + c, \quad \mathcal{B} = \rho_{ij}D_iD_j + \sigma_iD_i + \nu.$$

Definition 3.9. A solution u of (3.25) is an element of $\bigcup_{p,q} (\mathfrak{L})_{p,q}(\mathbb{W}; L_2((0, T); H^1(\mathbb{R}^d)))$ such that, for all sufficiently small h and all $t \in [0, T]$, the equality

$$(3.28) \quad u_h(t, x) = v_h(x) + \int_0^t (\mathcal{A} + h(s)\mathcal{B})u_h(s, x)ds$$

holds in $H^{-1}(\mathbb{R}^d)$.

The following theorem is the main result of this paper.

Theorem 3.10. Assume that, for some $p > 0$ and $q > 1$, $u_0 \in (\mathfrak{L})_{p,q}(\mathbb{W}; L_2(\mathbb{R}^d))$ and f, g are elements of the space $(\mathfrak{L})_{p,q}(\mathbb{W}; L_2((0, T); H^{-1}(\mathbb{R}^d)))$. Then there exist $r, \ell < 0$ such that equation (3.25) has a unique solution $u \in (\mathfrak{L})_{r,\ell}(\mathbb{W}; L_2((0, T); H^1(\mathbb{R}^d)))$ and

$$(3.29) \quad \int_0^T \|u(t)\|_{r,\ell;1}^2 dt \leq C \cdot \left(\|v\|_{p,q;0}^2 + \int_0^T \left(\|f(t)\|_{p,q;-1}^2 + \|g(t)\|_{p,q;-1}^2 \right) dt \right).$$

The number $C > 0$ depends only on $\delta, C_0, p, q, r, \ell$, and T .

Proof. The proof consists of two steps: first, we prove the result for deterministic functions v, f, g and then use linearity to extend the result to the general case.

Step 1. Assume that the functions $v \in L_2(\mathbb{R}^d)$, $f, g \in L_2((0, T); H^{-1}(\mathbb{R}^d))$ are deterministic. Then $v_h = v$, $f_h = f$, $g_h = g$, and classical theory of parabolic equations shows that, for sufficiently small h , equation (3.28) has a unique solution u_h and the dependence of u_h on h is analytic.

As in the previous section, we write

$$(3.30) \quad u(t, x) = \sum_{\alpha \in \mathcal{J}} u_\alpha(t, x) \xi_\alpha$$

where the coefficients u_α satisfy

$$\begin{aligned}
 u_{(0)}(t, x) &= v(x) + \int_0^t (\mathcal{A}u_{(0)}(s, x) + f(s, x))ds, \\
 (3.31) \quad u_{\epsilon_k}(t, x) &= \int_0^t \mathcal{A}u_{\epsilon_k}(s, x)ds + \int_0^t (\mathcal{B}u_{(0)}(s, x) + g(s, x))m_k(s)ds, \\
 u_\alpha(s, x) &= \int_0^s \mathcal{A}u_\alpha(s, x)ds + \sum_k \sqrt{\alpha_k} \int_0^s \mathcal{B}u_{\alpha-\epsilon_k}(s, x)m_k(s)ds, \quad |\alpha| > 1.
 \end{aligned}$$

Denote by $\Phi = \Phi_{s,t}$, $t \geq s \geq 0$ the semigroup generated by the operator \mathcal{A} . It follows by induction on $|\alpha|$ that

$$\begin{aligned}
 (3.32) \quad u_{(0)}(t, x) &= \Phi_{t,0}v(x) + \int_0^t \Phi_{t,s}f(s)ds, \\
 u_{\epsilon_k}(t, x) &= \int_0^t \Phi_{t,s}(\mathcal{B}u_{(0)}(s, x) + g(s, x))m_k(s)ds, \\
 u_\alpha(t, x) &= \frac{1}{\sqrt{\alpha!}} \int_0^t \int_0^{s_n} \dots \int_0^{s_2} \Phi_{t,s_n} \mathcal{B}\Phi_{s_n,s_{n-1}} \dots \mathcal{B}\Phi_{s_2,s_1} (\mathcal{B}u_{(0)}(s_1, x) + g(s_1, x)) \\
 &\quad E_\alpha(s_1, \dots, s_n) ds_1 \dots ds_n, \quad |\alpha| = n > 1.
 \end{aligned}$$

Therefore, using the usual parabolic estimates,

$$(3.33) \quad \int_0^T \|u_\alpha(t)\|_{H^1(\mathbb{R}^d)}^2 dt \leq \frac{C^n n!}{\alpha!} \left(\|v\|_{L_2(\mathbb{R}^d)}^2 + \int_0^T \left(\|f(t)\|_{H^{-1}(\mathbb{R}^d)}^2 + \|g(t)\|_{H^{-1}(\mathbb{R}^d)}^2 \right) dt \right),$$

and then (3.29) follows from (3.16).

Step 2. As in Step 1, existence and uniqueness of solution follows from unique solvability of the parabolic equation (3.28), and it remains to establish (3.29).

Denote by $u(t, x; V, F, G, \gamma)$, $\gamma \in \mathcal{J}$, the solution of (3.25) with $v = V\xi_\gamma$, $f = F\xi_\gamma$, $g = G\xi_\gamma$. If $v = \sum_{\alpha \in \mathcal{J}} v_\alpha \xi_\alpha$, etc., then

$$(3.34) \quad u(t, x) = \sum_{\gamma \in \mathcal{J}} u(t, x; v_\gamma, f_\gamma, g_\gamma, \gamma).$$

It follows from (3.31) that $u_\alpha(t, x; V, F, G, \gamma) = 0$ if $|\alpha| < |\gamma|$ and

$$(3.35) \quad \frac{u_{\alpha+\gamma}(t, x; V, F, G, \gamma)}{\sqrt{(\alpha+\gamma)!}} = \frac{u_\alpha\left(t, x; \frac{V}{\sqrt{\gamma!}}, \frac{F}{\sqrt{\gamma!}}, \frac{G}{\sqrt{\gamma!}}, (0)\right)}{\sqrt{\alpha!}}.$$

Using the results of Step 1,

$$\begin{aligned}
 (3.36) \quad &\int_0^T \|u(t, \cdot; v_\gamma, f_\gamma, g_\gamma)\|_{r,\ell;1}^2 dt \\
 &\leq \frac{C}{\gamma!} \left(\|v_\gamma\|_{L_2(\mathbb{R}^d)}^2 + \int_0^T \left(\|f_\gamma(t)\|_{H^{-1}(\mathbb{R}^d)}^2 + \|g_\gamma(t)\|_{H^{-1}(\mathbb{R}^d)}^2 \right) dt \right).
 \end{aligned}$$

Now (3.29) follows from (3.34) by the triangle inequality. \square

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Current address, S. V. Lototsky: Department of Mathematics, USC, Los Angeles, CA 90089

E-mail address, S. V. Lototsky: lototsky@math.usc.edu

URL: <http://www-rcf.usc.edu/~lototsky>

Current address, B. L. Rozovskii: Division of Applied Mathematics, Brown University, Providence, RI 02912

E-mail address, B. L. Rozovskii: rozovsky@dam.brown.edu